

On the singular p -Laplacian system under Navier slip type boundary conditions. The gradient-symmetric case.

H. Beirão da Veiga *

December 27, 2012

Abstract

We consider the p -Laplacian system of N equations in n space variables, $1 < p \leq 2$, under the homogeneous Navier slip boundary condition. Furthermore, the gradient of the velocity is replaced by the, more physical, symmetric gradient. We prove $W^{2,q}$ regularity, up to the boundary, under suitable assumptions on the couple p, q . The singular case $\mu = 0$ is covered.

Keywords: Singular p -Laplacian elliptic systems, slip boundary conditions, regularity up to the boundary.

1 Introduction. The main result

We consider the system

$$(1.1) \quad -\nabla \cdot ((\mu + |Du|^2)^{\frac{p-2}{2}} Du) = f \quad \text{in } \Omega$$

under the Navier slip boundary condition (1.3). Here, and in the sequel, u is an N -dimensional vector field defined in a bounded, open, connected, subset Ω of \mathbb{R}^n , locally situated on one side of its boundary, a smooth manifold Γ . We denote by \underline{n} the outer unit normal to $\partial\Omega$ and by $\mu \geq 0$ a given parameter. The vector field f is given. For convenience, we will assume that Ω has not axis of symmetry. The reason will be clear below. We are particularly interested in the singular case $\mu = 0$.

By

$$Du = \nabla u + \nabla^T u$$

we denote the symmetric gradient. So

$$(1.2) \quad D_{ij}(u) = \partial_i u_j + \partial_j u_i,$$

*Dipartimento di Matematica, Università di Pisa, Via Buonarroti 1/C, 56127 Pisa, Italy, email: bveiga@dma.unipi.it

where $i, j = 1, 2, \dots, n$. Often we simply write D , provided that the vector field under consideration follows from the context.

Equation (1.1) has been considered by mathematicians mostly with Du replaced by ∇u . It is worth noting that (1.1) satisfies the Stokes Principle (see [29], and [24] page 231), a significant physical requirement of isotropy, which does not hold if we replace Du by ∇u .

In the following we denote by $\underline{t}(u)$ the Cauchy stress vector

$$\underline{t}(u) = (Du) \cdot \underline{n}.$$

So,

$$t_j = (\partial_i u_j + \partial_j u_i) n_i,$$

where (here and in the sequel) we use the summation convention on repeated indexes.

The homogeneous Navier slip type boundary condition, see [22], says that the velocity is tangent to the boundary, and the tangential component of the stress vector $\underline{t}(u)$ vanishes on the boundary. We write this condition in the following form

$$(1.3) \quad \begin{cases} \underline{u} \cdot \underline{n} = 0, \\ (\underline{t}(u))_\tau = 0, \end{cases}$$

where in general the subscript τ denotes tangential component. For a mathematical study of the above boundary condition see, for instance, [6] and [28], where this boundary condition is associated to the linear Stokes problem.

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, m nonnegative integer, we denote the usual Lebesgue and Sobolev spaces, with the standard norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively.

In notation concerning norms and functional spaces, we do not distinguish between scalar and vector fields. For instance $L^p(\Omega; \mathbb{R}^N) = [L^p(\Omega)]^N$, $N > 1$, is denoted simply by $L^p(\Omega)$. We define

$$V_p = V_p(\Omega) = \{v \in W^{1,p}(\Omega) : v \cdot \underline{n} = 0 \text{ on } \Gamma\}.$$

The linear space $V_p(\Omega)$, endowed with one of the following norms

$$(\|v\|_p + \|Dv\|_p)^{\frac{1}{p}}, \quad (\|v\|_p + \|\nabla v\|_p)^{\frac{1}{p}}, \quad \|\nabla v\|_p,$$

is a Banach space. The above norms are equivalent in $V_p(\Omega)$. Further, since we assume that the domain Ω has not axis of symmetry, it follows that $\|Dv\|_p$ alone is a norm. For a quite complete discussion on this point, we refer to [6]. Without this hypothesis, existence or uniqueness of the solution may fail, depending on the particular external force f . We believe that this should be not difficult to show, by appealing to counter examples. However, a complete study of the possible phenomena (due to nonlinearity) should be difficult but quite interesting. Note that, if we replace Du by ∇u , the above assumption on Ω is superfluous.

For the proof of Korn's inequality we refer, for instance, to [28] and [23].

Our main result is the following.

Theorem 1.1. *Assume that $\mu \geq 0$, and let $f \in L^q(\Omega)$, where $q > n$. Let $C_q = C(q, \Omega)$ be the constant that appears in the linear estimate (3.5) below. Assume that*

$$(1.4) \quad (2 - p) C_q < 1.$$

Then, the weak solution u to the problem (1.1), (1.3) belongs to $W^{2,q}(\Omega)$. Moreover, the following estimate holds

$$(1.5) \quad \|u\|_{2,q} \leq C \left(\|f\|_q + \|f\|_q^{\frac{1}{p-1}} \right).$$

The proofs also apply, in a simpler way, to the Dirichlet boundary value problem. In section 7 we consider the boundary value problem (7.1). The singular case remains open. Finally, we refer to section 8 for an application to the fluid mechanics system (8.4).

Regularity of solutions for systems like (1.1) has received substantial attention from many authors. We refer, for instance, to references [1], [16], [18], [20], [30], [31]. Other related results may be found in [3], [4], [10], [12], [13], [15], [21], and references therein.

The plan of the paper is the following: In section 2 we recall the existence and uniqueness result of the weak solution. In section 3 we introduce an auxiliary linear problem and state (by appealing to well know classical results) the existence of solutions to this linear problem in spaces $W^{2,q}(\Omega)$. In section 3 we formulate the non-linear problem in a more explicit, formally equivalent, form in which the non-linearities are (roughly speaking) concentrated in the right hand side (see equation (4.3) below). Furthermore, we appeal to this formulation to define "strong solution". In section 5, by assuming $\mu > 0$ and by appealing to the result stated in section 3 for the auxiliary linear problem, we show that the strong solution introduced in section 5 exists and belongs to $W^{2,q}(\Omega)$, for each $\mu > 0$. Moreover, the estimates obtained are independent of μ . This last property allows us to extend, in section 6, the regularity result to the singular case $\mu = 0$ by passing to the limit in the variational formulation (2.4) as μ tends to zero. In section 7 we consider the boundary value problem (7.1). Finally, in section 8, we appeal to a recent result proved by Petr Kaplický and Jakub Tichý, to show that the result claimed in theorem 1.1 applies to solutions to the system (8.4), in the particular case $q = \hat{q}$, see (8.2).

2 Existence and uniqueness of the weak solution

Existence and uniqueness of weak solutions follows from well know results. Let us recall some basic points. Set

$$(2.1) \quad B(Du) = (\mu + |Du|^2)^{\frac{p-2}{2}}.$$

By appealing to the identity $D_{ij}u D_{ij}v = 2 D_{ij}u \partial_j v_i$, integration by parts shows that

$$(2.2) \quad \begin{aligned} \frac{1}{2} \int_{\Omega} B(Du) \cdot Du \cdot Dv \, dx &= - \int_{\Omega} \nabla \cdot (B(Du) D(u)) \cdot v \, dx \\ &+ \int_{\Gamma} B(Du) [(Du) \cdot v \cdot n] \, dS. \end{aligned}$$

Hence,

$$(2.3) \quad \begin{aligned} \frac{1}{2} \int_{\Omega} B(Du) \cdot Du \cdot Dv \, dx &= - \int_{\Omega} \nabla \cdot (B(Du) D(u)) \cdot v \, dx \\ &+ \int_{\Gamma} B(Du) (\underline{t}(u))_{\tau} \cdot v \, dS, \end{aligned}$$

provided that $v \cdot n = 0$ on Γ .

This last identity justifies the following definition.

Definition 2.1. *Let $f \in V_p'(\Omega)$. We say that u is a weak solution of problem (1.1), (1.3) if $u \in V_p(\Omega)$ satisfies*

$$(2.4) \quad \frac{1}{2} \int_{\Omega} B(Du) \cdot Du \cdot Dv \, dx = \int_{\Omega} f \cdot v \, dx,$$

for all $v \in V_p(\Omega)$.

Existence and uniqueness of the above weak solution, for any fixed $\mu \geq 0$, follows by appealing to the theory of monotone operators, see J.-L. Lions [19].

3 An auxiliary linear problem

In this section we consider the linear problem

$$(3.1) \quad -\nabla \cdot (Du) = F \text{ in } \Omega$$

under the boundary condition (1.3), and state an auxiliary result to be used in the next sections. This particular result follows from well known general results.

Note that equation (3.1) may be also written in the equivalent form (not used in this section)

$$(3.2) \quad -\Delta u - \nabla(\nabla \cdot u) = F.$$

Definition 3.1. *Let $f \in V_2'(\Omega)$. We say that u is a weak solution of problem (3.1), (1.3) if $u \in V_2(\Omega)$ satisfies*

$$(3.3) \quad \frac{1}{2} \int_{\Omega} Du \cdot Dv \, dx = \int_{\Omega} F \cdot v \, dx,$$

for all $v \in V_2(\Omega)$.

Coerciveness of the bilinear form on the left hand side of (3.3) follows here by appealing to the fact that $\|Dv\|$ alone is a norm in $V_2(\Omega)$, since we have assumed that Ω has not axis of symmetry. Hence, existence, uniqueness, and the standard estimate holds for the above problem.

Next we consider the regularity of the solutions to the above linear system (3.1) (or, equivalently, (3.2)) under the boundary condition (1.3). The $W^{2,2}(\Omega)$ regularity may be proved, for instance, by following [28] and [6]. The reader may easily adapt the argument developed in [28], section 4. Further, as claimed in reference [28] section 4, by appealing to results proved in reference [25] (see

also [2]), the $W^{2,q}(\Omega)$ regularity, for arbitrarily large exponents q , follows. Actually, under suitable, canonical, regularity assumptions on F and Ω , $W^{m,q}(\Omega)$ regularity for arbitrarily large values of $m \geq 2$ follows.

Alternatively, we may follow [27] to show that the system (3.2), (1.3) is of Petrovskiĭ type (a subclass of Agmon-Douglis-Nirenberg elliptic systems). See, in particular, the Theorem 5.1 in reference [27]. This allows a simplified integral representation formula for the solutions to the above linear problem. Moreover, for Petrovskiĭ's systems, the $W^{2,2}$ -regularity yields full $W^{m,q}$ -regularity, provided that the data are sufficiently smooth. In particular, there is a constant \tilde{C}_q such that

$$(3.4) \quad \|u\|_{2,q} \leq \tilde{C}_q \|F\|_q.$$

Summarizing, the following result holds.

Theorem 3.1. *Consider the linear boundary value problem (3.1), (1.3). Assume that $F \in L^q(\Omega)$, for some $q \geq 2$. Then, the solution u to the above linear problem belongs to $W^{2,q}(\Omega)$. Furthermore, there is a constant $C_q = C_q(q, \Omega)$, such that*

$$(3.5) \quad \|\nabla Du\|_q \leq C_q \|F\|_q.$$

Clearly, $C_q \leq \tilde{C}_q$. The pointwise estimate

$$(3.6) \quad |\nabla^2 u| \leq 3 |\nabla Du| \leq 6 |\nabla^2 u|$$

shows that $\|u\|_{2,q}$ and $\|\nabla Du\|_q$ are equivalent norms in $W^{2,q}(\Omega) \cap V_q(\Omega)$.

Under the homogeneous Dirichlet boundary value problem the constant C_q is bounded from above by a constant K times q . Clearly, this nice behavior can not hold, with the same constant K , for arbitrarily large values q . To each upper-bounded interval of values q it corresponds a distinct value K . See [32]. We do not know whether a similar (quite predictable) result is known for the Navier boundary condition.

4 The strong solution. Definition.

The main lines followed in this section have their starting point in some ideas already used, in a more complex context, in reference [7] (see, for instance, equations (4.17), (4.25), (4.26), and (4.27) in this last reference). Since

$$\nabla (\mu + |Du|^2)^{\frac{p-2}{2}} = \frac{p-2}{2} (\mu + |Du|^2)^{\frac{p-4}{2}} \nabla (|Du|^2),$$

straightforward calculations show that

$$(4.1) \quad \begin{aligned} \nabla \cdot ((\mu + |Du|^2)^{\frac{p-2}{2}} Du) &= (\mu + |Du|^2)^{\frac{p-2}{2}} \nabla \cdot (Du) \\ &+ (p-2) (\mu + |Du|^2)^{\frac{p-4}{2}} I(u) \end{aligned}$$

where, by definition,

$$I(u) = \frac{1}{2} \nabla (|Du|^2) \cdot Du = (Du : \nabla Du) \cdot Du.$$

The j component of the vector field $I(u)$ is given by

$$I_j(u) = \sum_k \sum_{l,m} D_{lm} (\partial_k D_{lm}) D_{kj}.$$

By improving an argument already used in [7], we may prove (as in the proof of Lemma 3.4 in [8]) the algebraic relation

$$|I \cdot \xi| \leq |D|^2 |\nabla D u| |\xi|,$$

for each arbitrary vector field in $\xi \in \mathbb{R}^N$, where

$$|\nabla D u|^2 = \sum_{m,l,k} (\partial_k D_{ml})^2.$$

Consequently, the pointwise estimate

$$(4.2) \quad |I(u)| \leq |D|^2 |\nabla D u|$$

holds.

Next we introduce the notion of strong solution used in the next section.

Definition 4.1. Assume that $\mu > 0$, and let $f \in L^q(\Omega)$ be given, $q > 1$. We say that $u \in W^{2,q}(\Omega)$ is a strong solution of problem (1.1), (1.3) if u satisfies (1.3) in the trace sense and, moreover, the equation

$$(4.3) \quad -\nabla \cdot (D u) = (p-2)G(u) + (\mu + |D u|^2)^{\frac{p-2}{2}} f$$

holds almost everywhere in Ω , where

$$G(v) = (\mu + |D v|^2)^{-1} I(v).$$

Note that $G(v) \leq |\nabla D v|$, almost everywhere in Ω , for all μ . So,

$$(4.4) \quad \|G(v)\|_q \leq \|\nabla D v\|_q.$$

5 Existence of the strong solution for $\mu > 0$.

Fix $\mu > 0$, and let $f \in L^q(\Omega)$. Following [11], by appealing to a fixed point argument, one proves the existence of a (unique) strong solution $u \in W^{2,q}(\Omega)$ of the above problem. Let us sketch the proof.

Since $q > n$, there is a constant $\widehat{C}(q, \Omega)$ such that

$$(5.1) \quad \|D v\|_\infty \leq \widehat{C} \|\nabla D v\|_q,$$

for all $v \in W^{2,q}(\Omega) \cap V_q(\Omega)$. Hence,

$$(5.2) \quad \| |D v|^{2-p} f \|_q \leq \|D v\|_\infty^{2-p} \|f\|_q \leq \widehat{C}^{2-p} \|\nabla D v\|_q^{2-p} \|f\|_q.$$

Further, since $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for nonnegative a and b , and $0 < \alpha < 1$, it follows that

$$(5.3) \quad (\mu + |D v|^2)^{\frac{2-p}{2}} \leq \mu^{\frac{2-p}{2}} + |D v|^{2-p}.$$

From (5.2) and (5.3) we show that

$$(5.4) \quad \|(\mu + |Dv|^2)^{\frac{2-p}{2}} f\|_q \leq \mu^{\frac{2-p}{2}} \|f\|_q + \widehat{C}^{2-p} \|\nabla Dv\|_q^{2-p} \|f\|_q.$$

Next we define the convex closed set

$$(5.5) \quad \mathbb{K} = \mathbb{K}(R) = \{v \in W^{2,q}(\Omega) \cap V_q(\Omega) : \|\nabla Dv\|_q \leq R\},$$

and consider, for each $v \in \mathbb{K}$, the solution $u = T(v)$ to the problem

$$(5.6) \quad -\nabla \cdot Du = F(v) \equiv (p-2)G(v) + (\mu + |Dv|^2)^{\frac{2-p}{2}} f,$$

under the boundary conditions (1.3).

By appealing to equations (3.5), (4.4), and (5.4), we obtain the estimate

$$(5.7) \quad \|\nabla Du\|_q \leq C_q \{(2-p)\|\nabla Dv\|_q + \mu^{\frac{2-p}{2}} \|f\|_q + \widehat{C}^{2-p} \|\nabla Dv\|_q^{2-p} \|f\|_q\}.$$

Next we show that if $\|\nabla Dv\|_q \leq R$ then the corresponding solution $u = T(v)$ satisfies the same estimate, namely $\|\nabla Du\|_q \leq R$. This shows that $T(\mathbb{K}) \subset \mathbb{K}$.

Since $v \in \mathbb{K}$ it follows that

$$(5.8) \quad \|\nabla Du\|_q \leq \mu^{2-p} C_q \|f\|_q + (2-p) C_q R + C_q \widehat{C}^{2-p} \|f\|_q R^{2-p}.$$

By assuming (1.4), we show that $u \in \mathbb{K}(R)$ if

$$[1 - (2-p) C_q] R \geq \mu^{2-p} C_q \|f\|_q + C_q \widehat{C}^{2-p} \|f\|_q R^{2-p}.$$

This inequality is satisfied if, for instance, its left hand side is equal to two times the sum of the two terms on the right hand side. This holds for

$$(5.9) \quad R = \frac{2}{\alpha} \mu^{2-p} C_q \|f\|_q + \left(\frac{2 C_q \widehat{C}^{2-p}}{\alpha} \right)^{\frac{1}{p-1}} \|f\|_q^{\frac{1}{p-1}},$$

where $\alpha = 1 - C_2(q)(2-p)$. Hence $\|\nabla Du\|_q \leq R$, and the inclusion $T(\mathbb{K}) \subset \mathbb{K}$ follows. This is the main ingredient to prove the existence of a fixed point in \mathbb{K} . For the missing details we refer to the argument developed in reference [11].

The expression of R shows that the uniform estimate (1.5) follows. Actually, we have shown that, for each positive μ , the estimate

$$(5.10) \quad \|u^\mu\|_{2,q} \leq C \left(\|f\|_q + \|f\|_q^{\frac{1}{p-1}} \right)$$

holds, where u^μ denotes the strong solution related to the particular positive value μ .

6 Existence of the strong solution for $\mu = 0$.

In this section, since the estimate (5.10) is uniform with respect to values μ (assumed to be bounded from above), by appealing to a compactness argument, we pass to the limit, as μ tends to zero, in the weak formulation (2.4) (which contains the singular case $\mu = 0$) and prove that the weak solution u to the singular problem also belongs to $W^{2,q}(\Omega)$, and satisfies (1.5).

We start by recalling the definition of weak solution u^μ of problem (1.1), for $\mu \geq 0$:

$$(6.1) \quad \int_{\Omega} (\mu + |Du^\mu|^2)^{\frac{p-2}{2}} Du^\mu \cdot Dv \, dx = \int_{\Omega} f \cdot v \, dx,$$

for all $v \in V_p(\Omega)$. This condition is satisfied by the strong solutions u^μ , for $\mu > 0$, constructed in the previous section. Since these solutions are uniformly bounded in $W^{2,q}(\Omega)$, suitable sub-sequences, which we continue to denote by u^μ , weakly converge in $W^{2,q}(\Omega)$ to some u . The argument followed in [11] shows that we may pass to the limit in (6.1) to prove that

$$(6.2) \quad \int_{\Omega} |Du|^{\frac{p-2}{2}} Du \cdot Dv \, dx = \int_{\Omega} f \cdot v \, dx,$$

for all $v \in V_p(\Omega)$. So, $u \in W^{2,q}(\Omega)$ is the solution (known to be unique), corresponding to $\mu = 0$. To prove the above claim, we have to show that, for each fixed $v \in V_p(\Omega)$, the left hand side of equation (6.1) converges to the left hand side of (6.2). Essentially, the proof followed in reference [11] section 4 applies here. For the reader's convenience, we repeat the main argument here.

Since $u^\mu \rightharpoonup u$ weakly in $W^{2,q}(\Omega)$, and $q > n$, strong convergence (of suitable subsequences) in $W^{1,s}(\Omega)$, for any s , follows. So, strong convergence in $W^{1,p}(\Omega)$ holds.

We write the integral on the left-hand side of (6.1) as

$$(6.3) \quad \int_{\Omega} [(\mu + |Du^\mu|^2)^{\frac{p-2}{2}} Du^\mu - (\mu + |Du|^2)^{\frac{p-2}{2}} Du] \cdot Dv \, dx \\ + \int_{\Omega} (\mu + |Du|^2)^{\frac{p-2}{2}} Du \cdot Dv \, dx,$$

and show that the first integral tends to zero, and the second integral tends to the left hand side of (6.2). The inequality

$$(6.4) \quad |(\mu + |A|)^{\frac{p-2}{2}} A - (\mu + |B|)^{\frac{p-2}{2}} B| \leq C \frac{|A - B|}{(\mu + |A| + |B|)^{2-p}},$$

where C is independent of μ (see [14] equation (6.8)), shows that the absolute value of the first integral in equation (6.3) is bounded by

$$C \int_{\Omega} (\mu + |Du| + |Du^\mu|)^{p-2} |Du - Du^\mu| |Dv| \, dx.$$

Since

$$(\mu + |Du| + |Du^\mu|)^{p-2} |Du - Du^\mu| \leq |Du - Du^\mu|^{p-1},$$

the absolute value of the first integral in equation (6.3) is bounded by

$$C \|Du^\mu - Du\|_p^{p-1} \|Dv\|_p,$$

which tends to zero with μ .

Finally,

$$\lim_{\mu \rightarrow 0^+} \int_{\Omega} (\mu + |Du|^2)^{\frac{p-2}{2}} Du \cdot Dv \, dx = \int_{\Omega} |Du|^{p-2} Du \cdot Dv \, dx,$$

by Lebesgue's dominated convergence theorem.

7 On a related slip boundary condition

In this section we consider the system (1.1) under the slip boundary condition

$$(7.1) \quad \begin{cases} \underline{u} \cdot \underline{n} = 0, \\ \underline{\omega}(u) \times \underline{n} = 0, \text{ on } \Gamma, \end{cases}$$

where $\underline{\omega} = \underline{\omega}(u) = \text{curl } \underline{u}$, and $n = N = 3$. The boundary condition (7.1), introduced by C. Bardos in reference [5], has been studied by a large number of authors.

In the following preliminary approach to the above problem, we start by a sketch of the proof of the existence of the strong solution, under the assumption $\mu > 0$. Moreover, an uniform $W^{2,q}(\Omega)$ estimate is claimed. However, the case $\mu = 0$ is not considered here, due to the lack of a suitable definition of weak solution. This point will be discussed below.

Concerning the existence of a strong solution for each $\mu > 0$, by taking into account definition 4.1, and by appealing to (1.1) and (4.1), we say that u is a strong solution of problem (1.1), (7.1) if $u \in W^{2,2}(\Omega)$ enjoys the boundary condition (7.1), and satisfies equation (4.3) almost everywhere in Ω . We write here the equation (4.3) in the equivalent form

$$(7.2) \quad \begin{aligned} -\Delta u - \nabla(\nabla \cdot u) = \\ (p-2)(\mu + |Du|^2)^{-1} I(u) + (\mu + |Du|^2)^{\frac{p-2}{2}} f. \end{aligned}$$

Let us prove the following result.

Lemma 7.1. *The following identity holds.*

$$(7.3) \quad \begin{aligned} \int_{\Omega} \nabla(\nabla \cdot u) \cdot \Delta v \, dx = \\ \int_{\Omega} \nabla(\nabla \cdot u) \cdot \nabla(\nabla \cdot v) \, dx - \int_{\Gamma} \nabla(\nabla \cdot u) \cdot (\underline{n} \times \underline{\omega}(v)) \, d\Gamma. \end{aligned}$$

In particular

$$(7.4) \quad \int_{\Omega} \nabla(\nabla \cdot u) \cdot \Delta u \, dx = \int_{\Omega} |\nabla(\nabla \cdot u)|^2 \, dx - \int_{\Gamma} \nabla(\nabla \cdot u) \cdot (\underline{n} \times \underline{\omega}(u)) \, d\Gamma.$$

Proof. Since

$$\Delta v = \nabla(\nabla \cdot v) - \nabla \times \underline{\omega}(v),$$

it follows that

$$(7.5) \quad \int_{\Omega} \nabla(\nabla \cdot u) \cdot \Delta v \, dx = \int_{\Omega} \nabla(\nabla \cdot u) \cdot \nabla(\nabla \cdot v) \, dx - \int_{\Omega} \nabla(\nabla \cdot u) \cdot \nabla \times \underline{\omega}(v) \, dx.$$

On the other hand, by appealing to the identity

$$\int_{\Omega} f \cdot (\nabla \times g) \, dx = \int_{\Omega} (\nabla \times f) \cdot g \, dx + \int_{\Gamma} f \cdot (\underline{n} \times g) \, d\Gamma,$$

we get

$$\int_{\Omega} \nabla (\nabla \cdot u) \cdot \nabla \times \underline{\omega}(v) dx = \int_{\Gamma} \nabla (\nabla \cdot u) \cdot (\underline{n} \times \underline{\omega}(v)) d\Gamma.$$

□

Note that the boundary integral in equation (7.4) vanishes if u satisfies the boundary condition $(\nabla \times u) \times \underline{n} = 0$ on Γ . So

$$(7.6) \quad \int_{\Omega} \nabla (\nabla \cdot u) \cdot \Delta u dx = \int_{\Omega} |\nabla (\nabla \cdot u)|^2 dx.$$

Multiplication of both sides of (7.2) by Δu , followed by integration in Ω , together with (7.6) and (4.2), leads to the following a priori estimate, uniform with respect to $\mu > 0$.

$$\begin{aligned} & \|\Delta u\|_2^2 + \|\nabla (\nabla \cdot u)\|_2^2 \leq \\ & (2-p) \int_{\Omega} |\nabla D u| \cdot \Delta u dx + \int_{\Omega} (\mu + |D u|^2)^{\frac{2-p}{2}} |f| |\Delta u| dx. \end{aligned}$$

This shows that, for each $\mu > 0$, it should be not difficult to prove the existence of a strong solution u in $W^{2,2}(\Omega)$, under the smallness assumption on $2-p$. By appealing to Agmon-Douglis-Nirenberg results, this should lead to an estimate of u in $W^{2,q}(\Omega)$, uniform with respect to $\mu > 0$. Clearly, we have to assume a restriction (like (1.4)) relying the exponents q and p . The estimates obtained are uniform with respect to $\mu > 0$. However, a suitable definition of weak solution, for $\mu \geq 0$, must be established, as a previous step to try to "pass to the limit" as μ goes to zero. Let us discuss this point.

We start by some identities. Since $(\partial_i u_k - \partial_k u_i) n_i = (\underline{\omega} \times \underline{n})_k$, the identity

$$(7.7) \quad (D u) \cdot \underline{n} \cdot v \equiv (\partial_k u_i + \partial_i u_k) n_i v_k = (\underline{\omega} \times \underline{n}) \cdot v + 2 (\partial_k u_i) v_k n_i$$

follows. Further, from

$$(7.8) \quad (\partial_k u_i) v_k n_i = \nabla(u \cdot \underline{n}) \cdot v - (\partial_k n_i) v_k u_i,$$

one gets

$$(7.9) \quad (D u) \cdot v \cdot n = (\underline{\omega} \times \underline{n}) \cdot v + 2 \nabla(u \cdot \underline{n}) \cdot v - 2 (\partial_k n_i) v_k u_i.$$

It follows, in particular, that on flat portions of the boundary the conditions (1.3) and (7.1) are equivalent.

By appealing to (2.2) and (7.9), and by assuming that $u \cdot n = v \cdot n = 0$ on Γ , we show that (recall definition (2.1))

$$(7.10) \quad \begin{aligned} & \frac{1}{2} \int_{\Omega} B(D u) \cdot D u \cdot D v dx = - \int_{\Omega} \nabla \cdot (B(D u) D(u)) \cdot v dx \\ & + \int_{\Gamma} B(D u) (\underline{\omega} \times \underline{n}) \cdot v dS - 2 \int_{\Gamma} B(D u) (\partial_k n_i) v_k u_i dS. \end{aligned}$$

The identity (7.10) would justify to call u a weak solution of problem (1.1), (1.3) if $u \in V_p(\Omega)$ satisfies

$$(7.11) \quad \begin{aligned} & \frac{1}{2} \int_{\Omega} B(Du) \cdot Du \cdot Dv \, dx + 2 \int_{\Gamma} B(Du) (\partial_k n_i) v_k u_i \, dS = \\ & = \int_{\Omega} f \cdot v \, dx, \end{aligned}$$

for all $v \in V_p(\Omega)$. Note that if the boundary integral in equation (7.11) vanishes for all v such that $v \cdot n = 0$, then $B(Du) (\underline{\omega} \times \underline{n}) = 0$ on Γ . Since $B(Du) \neq 0$, the second boundary condition (1.3) follows. However, the boundary integral in equation (7.11) is not well defined due to the term $B(Du)$, except for $p = 2$.

Note that, in equation (7.11), for $v = u$ and Ω convex, the integrand in the boundary integral is nonnegative. So, in a convex domain, we may obtain, at least, an a priori estimate in $W^{1,p}(\Omega)$.

8 On the Fluid Mechanics system

The proof of theorem 1.1 may be immediately adapted to the case $q < n$, also considered in [11] and [8]. In the case $q < n$, see [11], the assumption $f \in L^q(\Omega)$ does not imply $u \in W^{2,q}(\Omega)$. This last regularity result requires a stronger assumption on f , namely $f \in L^{r(q)}(\Omega)$, where $r(q)$ is given by

$$(8.1) \quad r(q) = \frac{nq}{n(p-1) + q(2-p)}.$$

Note that $r(q) > q$, and $r(n) = n$.

Since, on the whole, regularity results are stronger for large values of q , in [11] the authors have assumed, for convenience, that $q \geq 2$. This assumption excludes the significant case of square integrable external forces $f \in L^2(\Omega)$. In fact, by (8.1), $r(q) = 2$ holds for $q = \hat{q}$ given by

$$(8.2) \quad \hat{q} = \frac{2n(p-1)}{n-2(2-p)} < 2.$$

However, the proof shown in reference [11] also applies to values $q < 2$, in particular to \hat{q} . The (really obvious) modification required to adapt the proof to this case was shown in reference [9]. In this last reference we were interested in the particular case $r(q) = 2$, i.e. $q = \hat{q}$. In proposition 2.1 in [9] we basically remark that if $f \in L^2(\Omega)$ then u belongs to $W^{2,\hat{q}}(\Omega)$. Moreover,

$$(8.3) \quad \|u\|_{2,\hat{q}} \leq C \left(\|f\|_{\hat{q}} + \|f\|_2^{\frac{1}{p-1}} \right).$$

On the other hand, during a recent meeting in Levico (December 2012), Jakub Tichý informed us about some new results obtained in collaboration with Petr Kaplický, in reference [17]. The very interesting results obtained by these authors concern the generalized Stokes problem

$$(8.4) \quad \begin{cases} -\nabla \cdot ((\mu + |Du|^2)^{\frac{p-2}{2}} Du) + \nabla \pi = f, \\ \nabla \cdot u = 0, \end{cases}$$

under the Navier boundary condition (1.3). Actually they consider more general constitutive relations $S(D(u))$. In particular (see the theorem 1.8 in reference [17]), under natural assumptions on the external force f , the solution to problem (8.4), under the Navier boundary condition, satisfies

$$(8.5) \quad -\nabla \cdot ((\mu + |Du|^2)^{\frac{p-2}{2}} Du) \in L^2(\Omega).$$

So, by appealing to our theorem 1.1 (adapted, as described above, to the value $q = \hat{q}$), it follows that the solutions to problem (8.4) belong to $W^{2,\hat{q}}(\Omega)$. Clearly, we have to assume that condition (1.4) holds for the value $q = \hat{q}$.

We take the occasion to announce that some new results concerning the system (8.4) in the torus will be shown in a forthcoming paper.

References

- [1] E. Acerbi and N. Fusco, *Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$* , J. Math. Anal. Appl., **140** (1989), 115–135.
- [2] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math., **17** (1964), 35–92.
- [3] S. Antontsev and S. Shmarev, *Elliptic equations and systems with nonstandard growth conditions: Existence, uniqueness and localization properties for solutions*, Nonlinear Analysis, **65** (2006), 728–761.
- [4] S. Antontsev and S. Shmarev, *Anisotropic parabolic equations with variable nonlinearity*, Publ. Mat., **53** (2009), 355–399.
- [5] C. Bardos, *Existence et unicité de la solution de l'équation de Euler en dimension deux*, J. Math. Anal. Appl., **40** (1972), 769–790.
- [6] H. Beirão da Veiga, *Regularity for Stokes and generalized Stokes systems under non homogeneous slip type boundary conditions*, Advances in Differential Equations, **9**, (2004), no.9-10, 1079–1114.
- [7] H. Beirão da Veiga, *On the regularity of flows with Ladyzhenskaya shear dependent viscosity and slip or non-slip boundary conditions*, Comm. Pure Appl. Math. **58**, (2005), 552–577 (on line: 15.Jun.2004, doi: 10.1002/cpa.20036).
- [8] H. Beirão da Veiga, *On the global regularity for singular p -systems under non-homogeneous Dirichlet boundary conditions*, J. Math. Anal. Appl. **398** (2013), 527–533. doi: 10.1016/j.jmaa.2012.08.058
- [9] H. Beirão da Veiga, *On singular parabolic p -Laplacian systems under non-smooth external forces. Regularity up to the boundary*, submitted (Online arXiv:1206.1808v1 [math.AP], 8 Jun 2012).
- [10] H. Beirão da Veiga and F. Crispo, *On the global regularity for nonlinear systems of the p -Laplacian type*, to appear in *Discrete and Continuous Dynamical Systems - Series S*. Online arXiv:1008.3262v1 [math.AP].

- [11] H. Beirão da Veiga and F. Crispo, *On the global $W^{2,q}$ regularity for non-linear N -systems of the p -Laplacian type in n space variables*, Nonlinear Analysis-TMA , **75** (2012), 4346-4354. DOI: 10.1016/j.na.2012.03.021
- [12] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., **7** (1983), 827–850.
- [13] E. DiBenedetto and J. Manfredi, *On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems*, Amer. J. Math., **115** (1993), 1107–1134.
- [14] L. Diening, C. Ebmeyer and M. Růžička, *Optimal convergence for the implicit space-time discretization of parabolic systems with p -structure*, SIAM J. Numer. Anal., **45** (2007), 457–472.
- [15] M. Fuchs and G. Seregin, *Variational methods for problems from plasticity theory and for generalized Newtonian fluids*, Lecture Notes in Mathematics, 1749. Springer-Verlag, Berlin, 2000.
- [16] C. Hamburger, *Regularity of differential forms minimizing degenerate elliptic functionals*, J. Reine Angew. Math., **431** (1992), 7–64.
- [17] P.Kaplický and J. Tichý, *Boundary regularity of flows under perfect slip boundary conditions*, to appear in CEJM.
- [18] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., **12** (1988), 1203–1219.
- [19] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.
- [20] W. B. Liu and J. W. Barrett, *A remark on the regularity of the solutions of the p -Laplacian and its application to their finite element approximation*, J. Math. Anal. Appl., **178** (1993), 470–487.
- [21] P. Marcellini and G. Papi, *Nonlinear elliptic systems with general growth*, J. Differ. Equations, **221** (2006), 412–443.
- [22] C. L. M. Navier, *Mémoire sur les lois du mouvement des fluides*, *Mémoires de l'Académie Royale des Sciences de l'Institut de France*, 1 (1816).
- [23] C. Parés, *Existence, uniqueness and regularity of solutions of the equations of a turbulence model for incompressible fluids*, Appl. Analysis, **43** (1992), 245-296.
- [24] J. Serrin, *Mathematical Principles of Classical Fluid Mechanics*, in *Encyclopedia of Physics* VIII, p.p. 125-263, Springer-Verlag, Berlin, 1959.
- [25] V.A. Solonnikov, *General boundary value problems for Douglas-Nirenberg elliptic systems.II*, Proceedings of the Steklov Institute of Mathematics, **92** 1966.
- [26] V.A. Solonnikov, *On Green's Matrices for Elliptic Boundary Problem I*, Trudy Mat. Inst. Steklov **110** (1970), 123-170.

- [27] V.A. Solonnikov, *On Green's Matrices for Elliptic Boundary Problem II*, Trudy Mat. Inst. Steklov **116** (1971), 187-226.
- [28] V.A. Solonnikov and V.E. Ščadilov, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Proc. Steklov Inst. Math., **125** (1973), 186-199.
- [29] Stokes, G. Trans. Cambridge Phil. Soc., **8**, 287 (1845), 75-129.
- [30] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, **51** (1984), 126-150.
- [31] N.N. Ural'tseva, *Degenerate quasilinear elliptic systems*, Zap. NauK. Sem. LOMI, **7** (1968), 184-222 (in Russian).
- [32] V. I. Yudovic, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokl. Akad. Nauk SSSR, **138**(1961), 805-808, English translation in Soviet Math. Doklady **2** (1961), 746-749.